Math 246C Lecture 26 Notes

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1 L^2 Estimates for The $\overline{\partial}$ Operator in Several Complex Variables (cont.)

1.1 Conditions for an operator to be surjective

We have an operator $T : L^2(\Omega, e^{-\varphi_1}) \to L^2_{(0,1)}(\Omega, e^{-\varphi_2})$, acting as $\overline{\partial}$, where $\Omega \subseteq \mathbb{C}^n$ is open and $\varphi_1, \varphi_2 \in C^\infty(\Omega)$ are real weights to be chosen. Also $\operatorname{Ran}(T) \subseteq F = \{f \in L^2_{(0,1)}(\Omega, e^{-\varphi_2}) : \overline{\partial} f = 0\}$.

Lemma 1.1. *Let $T : H_1 \to H_2$ be linear, closed, and densely defined with $\operatorname{Ran}(T) \subseteq F$, where F is a closed subspace of H_2. Then $\operatorname{Ran}(T) = F$ if and only if there is a $C > 0$ such that $\|F\|_{H_2} \le C\|T^* f\|_{H_1}$ for all $f \in F \cap D(T^*)$.*

Proof. ($\implies$): Consider the map $T : D(T) \to F$, which are Banach spaces if $D(T)$ is equipped with the graph norm $\|u\|_{D(T)} := \|u\| + \|Tu\|$. T is continuous and surjective, so T is open by the open mapping theorem. Then $T(\{u : \|u\|_{D(T)} < 1\}) \supseteq \{f \in F : \|f\| < \varepsilon\}$ for some $\varepsilon > 0$. We get that there is a $C > 0$ such that for all $g \in F$, there is a $u \in D(T)$ such that $Tu = f$ and $\|u\|_{H_1} \le C\|g\|_{H_2}$. When $f \in D(T^*) \cap F$,

$$|\langle f, g \rangle_{H_2}| = |\langle f, Tu \rangle_{H_2}| = |\langle T^* f, u \rangle| \le C\|T^* f\|_{H_1}\|g\|_{H_2}.$$

We get that $\|f\|_{H_2} \le \|T^* f\|_{H_1}$.

($\impliedby$): Assume that the bound holds for all $f \in F \cap D(T^*)$. We have $\operatorname{Ran}(T) \subseteq F$. Let $g \in F$. We claim that the antilinear map $L(T^* f) = \langle f, g \rangle_{H_2}$ (for $f \in D(T^*)$) is well-defined and satisfies $|L(T^* f)| \le C\|g\|_{H_2}\|T^* f\|_{H_1}$.

We can write $f = f_! + f_2$, where $f_1 \in F$, and $f_2 \in F^\perp$ for any $f \in D(T^*)$. Now $\langle f_2, Tu \rangle = 0$ for any $u \in D(T)$, so $f_2 \in D(T^*)$; in particular, $T^* f_2 = 0$. So $f_1 \in F \cap D(T^*)$, and we get

$$|L(T^* f)| = \langle g, f_1 \rangle \le C\|g\|_{H_2}\|\underbrace{T^* f_1}_{=T^* f}\|_{H_1}.$$

So we get the claim.

We get that the map L extends by continuity to $\overline{\mathrm{Ran}(T^*)} \subseteq H_1$. So there is a $u \in \overline{\mathrm{Ran}(T^*)}$ such that $L(T^*f) = \langle u, T^*f \rangle_{H_1}$ for all $f \in D(T^*)$. On the other hand, $L(T^*f) := \langle g, f \rangle_{H_2}$, so we get $\langle T^*f, u \rangle = \langle f, g \rangle$ for all $f \in D(T^*)$. This implies that $u \in D((T^*)^*) = D(T)$ and $Tu = g$. We also get that

$$\|u\|_{H_1} = \|L\| \leq C\|g\|_{H_2}. \qquad \square$$

1.2 Hörmander's idea and the density lemma

In our setting $H_1 = L^2(\Omega, e^{-\varphi_1})$, $H_2 = L^2_{(0,1)}(\Omega, e^{-\varphi_2})$, $T = \overline{\partial}$, and $F = \{f \in H_2 : \overline{\partial} f = 0\}$. So we want to show that

$$\|f\|_{H_2} \leq C\|T^*f\|_{H_1}, \qquad f \in F \cap D(T^*).$$

Introduce the space of 2-forms

$$H_3 = L^2_{(0,2)}(\Omega, e^{-\varphi_3}) = F = \sum_{j,k} F_{j,k}\, d\overline{z}_j \wedge d\overline{z}_k : F_{j,k} \in L^2(\Omega, e^{-\varphi_3}),$$

and consider the closed, densely defined operator $S : H_2 \to H_3$ which sends $f \mapsto \overline{\partial} f = \sum_j \overline{\partial} f_j \wedge d\overline{z}_j = \sum_{j,k} \frac{\partial f_j}{\partial \overline{z}_k} d\overline{z}_k \wedge d\overline{z}_h$. We have $F = \ker(S)$. Rather than trying to prove the bound, we shall try to prove

$$\|f\|^2_{H_2} \leq C(\|T^*f\|^2_{H_1} + \|Sf\|^2_{H_3}), \qquad \forall f \in D(T^*) \cap D(S).$$

This looks stronger, but it has symmetry properties we can exploit.

The idea, due to Hörmander, is to choose the weights $\varphi_1, \varphi_2, \varphi_3$ so that the 1-forms with coefficients in $C_0^\infty(\Omega)$ are dense with respect to the graph norm $f \mapsto \|f\|_{H_2} + \|T^*f\|_{H_1} + \|Sf\|_{H_3}$.

Lemma 1.2 (Density lemma)**.** *Let (η_ν) be a sequence in $C_0^\infty(\Omega)$ such that $0 \leq \eta_\nu \leq 1$ and such that for any compact $K \subseteq \Omega$, $\eta_\nu = 1$ on K for all large ν. Assume that*

$$e^{-\varphi_{j+1}} |\overline{\partial} \eta_\nu|^2 \leq C e^{-\varphi_j}, \qquad \forall \nu, j = 1, 2.$$

Then $C^\infty_{0,(0,1)}(\Omega)$ is dense in $D(T^*) \cap D(S)$ with respect to the graph norm.

Remark 1.1. If $\Omega = \mathbb{C}^n$, we can take $\eta_\nu(z) = \eta(z/\nu)$ for some function η which is 1 near 0. Then we can take $\varphi_1 = \varphi_2 = \varphi_3$.

Proof. Step 1: Suppose $f \in D(T^*) \cap D(S)$ has compact support. Approximate by $f * \psi_\varepsilon$, where $\psi_\varepsilon(z) = \varepsilon^{-2n} \psi(z/\varepsilon)$ and $\psi \in C_0^\infty$.

Step 2: Given $f \in D(T^*) \cap D(S)$, consider $\eta_\nu f \in D(T^*) \cap D(S)$. Then $S(\eta_j f) \to Sf$ in H_3. Then

$$S(\eta_j f) = \underbrace{\eta_j Sf}_{L^2_{\varphi_3}} + \underbrace{[S, \eta_j]}_{=(\overline{\partial}\eta_j) f} \xrightarrow{L^2_{\varphi_3}} Sf$$

by dominated convergence. $\square$

We will review this last point in more detail next time.